International Journal of Theoretical Physics, Vol. 35, No. 12, 1996

Periodic Solutions and Perturbations of Dynamical Systems

Svetoslav Ivanov Nenov[1]

Received January 7, 1996

We deal with some problems concerning periodic solutions of perturbed dynamical systems. Sufficient conditions for the existence of periodic solution(s) of perturbed system are obtained. Moreover, we derive some properties of the set of all "perturbed" terms of a dynamical system under which the perturbed system has periodic solution(s). The method is based on the analysis of the space of all solutions of a nonperturbed dynamical system.

1. INTRODUCTION

The present paper deals with some problems concerning periodic solutions of perturbed dynamical systems.

Let $f \in \mathbf{C}^{\infty}(\mathbf{R}^n, \mathbf{R}^n)$. We consider the system

$$\dot{x} = f(x), \qquad x \in \mathbf{R}^n \tag{1}$$

and its perturbed analog

$$\dot{x} = f(x) + g(x, \epsilon), \qquad \epsilon \in \mathbf{R}^m \tag{2}$$

where $g \in \mathbf{C}^{\infty}(\mathbf{R}^n \times \mathbf{R}^m, \mathbf{R}^n)$, $g(x, 0) = 0$. Let $x = \bar{x}(t)$ be a nontrivial periodic solution of the nonperturbed system (1).

The classical statement of the problem of the existence of periodic solutions of perturbed systems is to determine conditions such that for all "sup-norm small" functions g the system (2) has periodic solution (Coddington and Levinson, 1955; Massera, 1950; Rouche *et al.*, 1977; Nemistsky and Stepanov, 1949; Yoshizawa, 1966) [for a detailed survey see Li (1981)]. Most results

[1] Department of Mathematics, University of Chemical Technology and Metallurgy, Sofia 1756, Bulgaria.

0020-7748/96/1200-2717$09.50/0

in the cited books and papers are based on the investigation of the Liapunov function for the system (1).

In the present paper, we assume that both functions f and g are given. We shall consider the following problems:

- The existence of periodic solution(s) of the perturbed system (2). We obtain the "maximal" subspace $\mathbf{R}^d \subset \mathbf{R}^m$ such that if $\epsilon \in \mathbf{R}^m$, $\epsilon = (\epsilon_1, \epsilon_2)$, $\epsilon_1 \in \mathbf{R}^d$, $\epsilon_2 = 0$, then the perturbed system (2) possesses periodic solution(s).
- The "description" of the set of all functions g for which the perturbed system (2) has periodic solution(s).
- The "maximal" deviation (from zero) of the function g such that the perturbed system (2) has periodic solution(s).

Some analogous results are introduced in Yoshizawa (1966, Chapter VI, §§25, 29).

The paper consists of four sections. In Section 2 we investigate "small parametric" extensions of the solutions of the equation $F(x, y) = 0$, where $F: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$; $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are Hilbert spaces and $F(0, 0) = 0$. In Lemma 1 we obtain conditions under which there exist smooth functions $f = f(v)$ and $g = g(v)$ such that $f(0) = 0$, $g(0) = 0$; $F(f(v), g(v)) = 0$, where v is a "parametric" vector with small enough $\|v\|$. Lemma 2 establishes a connection between the problem of "small parametric" extensions of the solution of the equation $F(x, y) = 0$ and Fredholmness of the operator $\mathbf{D}_x F(0, 0)$ (Kirilov and Gvishiany, 1979; Krein, 1967; Hutson and Pym, 1980).

Section 3 deals with our main results. Applying Lemmas 1 and 2, we prove the existence of periodic solutions of the perturbed system (2). Theorem 1 gives conditions for the existence of periodic solutions without a changing of the period [the period of the solution(s) of the perturbed system is the same as the period of the solution of the nonperturbed system]. In Theorem 2 we obtain conditions for the existence of periodic solutions with "small" change of the period. Some cases of specific perturbation of the system (1) and applications of Theorems 1 and 2 are considered.

2. A MODIFICATION OF IMPLICIT FUNCTION THEOREM

Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be Hilbert spaces. We shall use the following notation: $\mathcal{X} \oplus \mathcal{Y}$ denotes the direct sum of $\mathcal{X}$ and $\mathcal{Y}$. If $\mathcal{W}$ is a closed subspace of $\mathcal{X}$, then $\mathcal{W}^\perp$ denotes the orthogonal complement of $\mathcal{W}$. We set $\mathrm{B}_{\mathcal{X}}(x_0, r) = \{x \in \mathcal{X}: \|x - x_0\| < r\}$, where $x_0 \in \mathcal{X}$, $r > 0$, and $\|\cdot\|$ is the norm in $\mathcal{X}$.

Let $F: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$, $(x, y) \to F(x, y)$ be a $\mathbf{C}^\kappa$-smooth map, $\kappa \geq 1$. We shall denote by $\mathbf{D}_x F(x, y)$ $[\mathbf{D}_y F(x, y)]$ the derivative of F with respect to the first (second) argument. If $L: \mathscr{X} \to \mathscr{Y}$ is a linear operator, then ker L (im L) denotes the kernel (range) of L.

First, we shall prove the following lemma.

Lemma 1. Let the following conditions hold:

1. $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ are Hilbert spaces; $F: \mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$ is a $\mathbf{C}^1$-smooth map and $F(0, 0) = 0$.

2. There exists a closed subspace $\mathscr{W} \subseteq (\ker \mathbf{D}_y F(0, 0))^\perp$ such that

$$\dim \ker \mathbf{D}_x F(0, 0) = \dim \mathscr{W} = d < \infty \tag{3}$$

and

$$\mathscr{Z} = \operatorname{im} \mathbf{D}_x F(0, 0) \oplus \operatorname{im} \mathbf{D}_y F(0, 0)|_{\mathscr{W}} \tag{4}$$

3. There exists a number $M > 0$ such that

$$\|\mathbf{D}_\alpha F(x, v) - \mathbf{D}_\alpha F(x, 0)\| \le \frac{M}{2}\|v\|, \quad (x, v) \in \mathscr{X} \times \mathscr{W}^\perp, \quad \alpha \in \{x, y\} \tag{5}$$

Then:

1. There exist a number $r_1 > 0$ and unique $\mathbf{C}^1$-smooth maps $f: \mathbf{B}_{\mathscr{W}^\perp}(0, r_1) \to \mathscr{X}$ and $g: \mathbf{B}_{\mathscr{W}^\perp}(0, r_1) \to \mathscr{Y}$ such that $f(0) = 0$, $g(0) = 0$, and

$$F(f(v), g(v)) = 0 \qquad \text{for any} \quad v \in \mathbf{B}_{\mathscr{W}^\perp}(0, r_1) \tag{6}$$

2. If the operator $\mathbf{D}_y F(0, 0)|_{\mathscr{W}^\perp}$ is an isomorphism between the Hilbert spaces $\mathscr{W}^\perp$ and im $\mathbf{D}_y F(0, 0)|_{\mathscr{W}^\perp}$, then there exists $r_1' \in (0, r_1)$ such that the maps

$$f|_{\mathbf{B}_{\mathscr{W}^\perp}(0, r_1')} \qquad \text{and} \qquad g|_{\mathbf{B}_{\mathscr{W}^\perp}(0, r_1')}$$

are embeddings.

3. Let $\pi_x: \mathscr{X} \times \mathscr{Y} \to \mathscr{X}$ [$\pi_y: \mathscr{X} \times \mathscr{Y} \to \mathscr{Y}$] given by $\pi_x(x, y) = x$ [$\pi_y(x, y) = y$] be the projection on $\mathscr{X}(\mathscr{Y})$, and let $L(x, y)$ be a bounded linear operator defined by .

$$L(x, y): \ \mathscr{X} \times \mathscr{Y} \to \mathscr{Z}, \qquad L(x, y) = \mathbf{D}_x F(x, y) \circ \pi_x + \mathbf{D}_y F(x, y) \circ \pi_y \tag{7}$$

Let $L(0, 0)$ be an isomorphism, $\mathbf{1}_{\mathscr{X}\times\mathscr{Y}}$ be the identity on the space $\mathscr{X} \times \mathscr{Y}$, and the numbers $r_2, r_3 > 0$ be chosen such that

$$\|\mathbf{1}_{\mathscr{X}\times\mathscr{Y}} - L^{-1}(0, 0)L(x, 0)\| < 1/4 \qquad \text{for any} \quad x \in \mathbf{B}_{\mathscr{X}}(0, r_2) \tag{8}$$

$$r_3 M \|L^{-1}(0, 0)\| < 1/8 \tag{9}$$

$$\|L^{-1}(0, 0)\| \cdot \|L(0, y)\| < r_2/4 \qquad \text{for any} \quad y \in \mathbf{B}_{\mathscr{Y}}(0, r_3) \tag{10}$$

Then $r_1 \ge r_3$ and $(f(v), g(v)) \in \mathbf{B}_{\mathscr{X}\times\mathscr{Y}}(0, r_2)$ for any $v \in \mathbf{B}_{\mathscr{W}^\perp}(0, r_1)$.

Proof. Let $\mathcal{X}_1 = (\ker \mathbf{D}_x F(0, 0))^{\perp} \times \mathcal{W}$. We consider the space $\mathcal{X}_1 \times \mathcal{W}^{\perp}$.

If $(u, v) \in \mathcal{X}_1 \times \mathcal{W}^{\perp}$, then $u = (u', u'')$, where $u' \in (\ker \mathbf{D}_x F(0, 0))^{\perp}$ and $u'' \in \mathcal{W}$. From (3) it follows that the Hilbert spaces $\ker \mathbf{D}_x F(0, 0)$ and $\mathcal{W}$ are isomorphic: $\ker \mathbf{D}_x F(0, 0) \cong \mathcal{W}$. That is why we may assume $u'' \in \ker \mathbf{D}_x F(0, 0)$, i.e., $u \in \mathcal{X}$. On the other hand, the inclusions $u'' \in \mathcal{W}$ and $v \in \mathcal{W}^{\perp}$ yield $(u'', v) \in \mathcal{Y}$.

Therefore, each point $(u, v) \in \mathcal{X}_1 \times \mathcal{W}^{\perp}$ determines a point $(u, (u'', v)) \in \mathcal{X} \times \mathcal{Y}$. Let us consider the map $G: \mathcal{X}_1 \times \mathcal{W}^{\perp} \to \mathfrak{X}$ given by $G(u, v) = F(u, w)$, where $u = (u', u'') \in \mathcal{X}_1 \cong \mathcal{X}$ and $w = (u'', v) \in \mathcal{Y}$.

Some properties of the map G are as follows:

(a) $G = G(u, v)$ is a $\mathbf{C}^1$-smooth map and $G(0, 0) = 0$.

(b) $\mathbf{D}_u G(0, 0)u = \mathbf{D}_x F(0, 0)u + \mathbf{D}_y \mathbf{F}(0, 0)w_0$, where $u = (u', u'') \in \mathcal{X}_1$, $w_0 = (u'', 0) \in \mathcal{Y}$.

(c) The linear operator $\mathbf{D}_u G(0, 0): \mathcal{X}_1 \to \mathfrak{X}$ is an isomorphism.

Indeed, let $u \in \mathcal{X}_1 \backslash \{0\}$. We assume that $\mathbf{D}_u G(0, 0)u = 0$. From (b) it follows that $-\mathbf{D}_y F(0, 0)(u'', 0) \in \operatorname{im} \mathbf{D}_x F(0, 0)$ and $u'' \in \mathcal{W}$. On the other hand, formula (4) implies $\operatorname{im} \mathbf{D}_x F(0, 0) \cap \operatorname{im} \mathbf{D}_y F(0, 0)|_{\mathcal{W}} = \{0\}$. Hence, $u'' = 0$, i.e., $u = (u', 0)$. Then $\mathbf{D}_u G(0, 0)u = \mathbf{D}_x F(0, 0)(u', 0) \neq 0$, because of $u' \in (\ker \mathbf{D}_x F(0, 0))^{\perp}$.

This contradiction provides that the operator $\mathbf{D}_u G(0, 0)$ is an injection.

Let $z \in \mathfrak{X}$. From condition 3 of the lemma it follows that there exist unique points $z_1 \in \operatorname{im} \mathbf{D}_x F(0, 0)$ and $z_2 \in \operatorname{im} \mathbf{D}_y F(0, 0)|_{\mathcal{W}}$ such that $z = z_1 + z_2$. We choose the points $x \in \mathcal{X}$ and $u'' \in \mathcal{W}$ such that $z_1 = \mathbf{D}_x F(0, 0)x$, $z_2 = \mathbf{D}_y F(0, 0)w_0$, where $w_0 = (u'', 0)$. Then $z = \mathbf{D}_u G(0, 0)(x, y)$.

Thus $\mathbf{D}_u G(0, 0)$ is a surjection, i.e., $\mathbf{D}_u G(0, 0)$ is an isomorphism.

(d) If $(u, v) \in \mathcal{X}_1 \times \mathcal{W}^{\perp}$, then

$$\|\mathbf{D}_u G(u, v) - \mathbf{D}_u G(u, 0)\| \leq M\|v\|$$

Indeed (see condition 4 of the lemma)

$$\begin{aligned} &\|\mathbf{D}_u G(u, v) - \mathbf{D}_u G(u, 0)\| \\ &\quad \leq \|D_u F(u, v) - \mathbf{D}_u F(u, 0)\| + \|\mathbf{D}_v F(u, v) - \mathbf{D}_v F(u, 0)\| \\ &\quad \leq M\|v\| \end{aligned}$$

for any $(u, v) \in \mathcal{X}_1 \times \mathcal{W}^{\perp} \cong \mathcal{X} \times \mathcal{W}^{\perp}$.

Now we shall prove the assertions of Lemma 1.

1. From (a), (c), continuity of the operators $\mathbf{D}_x F(0, 0)$ and $\mathbf{D}_y F(0, 0)$, and the Implicit Functional Theorem (Hutson and Pym, 1980, Theorem 4.4.9) it follows that there exist $r_1 > 0$ and a unique function $h: \mathbf{B}_{\mathcal{W}^{\perp}}(0, r_1) \to \mathcal{X}_1$ such that $G(h(v), v) = 0$ for any $v \in \mathbf{B}_{\mathcal{W}^{\perp}}(0, r_1)$. Let $I: \mathcal{X}_1 \to \mathcal{X}$ be the

operator of isomorphism between the spaces $\mathcal{X}$ and $\mathcal{X}_1$. We set $f(v) = I \circ h(v)$ and $g(v) = (e \circ h(v), v)$, where $e: \mathcal{X}_1 \to \mathcal{W}$ is a projection. Then $F(f(v), g(v)) = G(h(v), v) = 0$ for any $v \in \mathbf{B}_{\mathcal{W}^\perp}(0, r_1)$.

2. The definition of the map h implies that $\mathbf{D}_u G(0, 0)\mathbf{D}_v h(0) + \mathbf{D}_v G(0, 0) = 0$. But the linear operator $\mathbf{D}_u G(0, 0)$ is an isomorphism [see (c)] and for any $v \in \mathcal{W}^\perp$, $\mathbf{D}_v G(0, 0)v = \mathbf{D}_y F(0, 0)\tilde{w}$, where $\tilde{w} = (0, v) \in \mathcal{Y}$. Then the linear operator

$$\mathbf{D}_v h(0) = -(\mathbf{D}_u G(0, 0))^{-1}\mathbf{D}_v G(0, 0): \quad \mathcal{W}^\perp \to \mathcal{X}$$

is an isomorphism, too.

Therefore, property 3 of the lemma is a result of the Inverse Mapping Theorem (Nitecki, 1971, Chapter 2, §1).

3. This follows similarly as in part 1, making use of Theorem 4.4.10 from Hutson and Pym (1980).

Lemma 1 is proved.

Before giving a property of map F equivalent to (4), we shall recall the definition of the Fredholm linear operator.

Let $L: \mathcal{X} \to \mathcal{Z}$ be a bounded linear operator and let coker $L = \mathcal{Z}/\text{im } L$ be the cokernel of L. The linear operator L is said to be *Fredholm with index zero* if dim ker $L < \infty$, dim coker $L < \infty$, and dim ker $L =$ dim coker L (Kirilov and Gvishiany, 1979; Krein, 1967; Hutson and Pym, 1980).

Lemma 2. Let the following conditions hold:

1. $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are Hilbert spaces, $F: \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is a $\mathbf{C}^1$-smooth map, and $F(0, 0) = 0$.

2. There exists a closed subspace $\mathcal{W} \subseteq (\ker \mathbf{D}_y F(0, 0))^\perp$ such that formula (3) is valid and let

$$\text{im } \mathbf{D}_x F(0, 0) \cap \text{im } \mathbf{D}_y F(0, 0)|_{\mathcal{W}} = \{0\} \tag{11}$$

Then equality (4) is valid if and only if $\mathbf{D}_x F(0, 0)$ is a Fredholm operator with index zero, i.e.,

$$\dim \ker \mathbf{D}_x F(0, 0) = \dim \text{coker } \mathbf{D}_x F(0, 0) \tag{12}$$

Proof. Let formula (4) be valid. Then

$$\begin{aligned} \dim \ker \mathbf{D}_x F(0, 0) &= \dim \mathcal{W} \\ &= \dim \text{im } \mathbf{D}_y F(0, 0)|_{\mathcal{W}} \\ &= \dim \text{coker } \mathbf{D}_x F(0, 0) \end{aligned}$$

due to the fact that $\mathbf{D}_y F(0, 0)|_{\mathcal{W}}$ is a nonsingular operator (see condition 2). Hence $\mathbf{D}_x F(0, 0)$ is a Fredholm operator with index zero.

Let $\mathbf{D}_x F(0, 0)$ be a Fredholm operator with index zero. First we shall prove that there exists a finite-dimensional subspace $\mathcal{R} \subset \mathfrak{X}$ such that $\mathfrak{X} = \mathcal{R} \oplus \text{im } \mathbf{D}_x F(0, 0)$.

Let $\{\tilde{z}_i: i \in \{1, \ldots, \beta\}\}$ be a basis for coker $\mathbf{D}_x F(0, 0)$; z_i represent the class $\tilde{z}_i$, $i \in \{1, \ldots, \beta\}$; $\mathcal{R}$ denotes the linear closure of $\{z_i: i \in \{1, \ldots, \beta\}\}$; $z \in \mathfrak{X}$ and $\tilde{z}$ is the corresponding vector in coker $\mathbf{D}_x F(0, 0)$. Then there exists a unique sequence $\{c_i: i \in \{1, \ldots, \beta\}\} \subset \mathbf{R}$ such that $\tilde{z} = \Sigma_{i=1}^{\beta} c_i \tilde{z}_i$. From the definition of factor space it follows that $z = \Sigma_{i=1}^{\beta} c_i z_i + t$, where $t \in \text{im } \mathbf{D}_x F(0, 0)$. Therefore, $\mathfrak{X} = \mathcal{R} \oplus \text{im } \mathbf{D}_x F(0, 0)$.

The equalities

$$\dim \mathcal{R} = \text{codim im } \mathbf{D}_x F(0, 0) = \dim \ker \mathbf{D}_x F(0, 0)$$
$$= \dim \mathcal{W} = \dim \text{im } \mathbf{D}_y F(0, 0)|_{\mathcal{W}}$$

yield that the spaces $\mathcal{R}$ and im $\mathbf{D}_y F(0, 0)|_{\mathcal{W}}$ are isomorphic. That is why the equality (11) implies $\mathcal{R} = \text{im } \mathbf{D}_y F(0, 0)|_{\mathcal{W}}$. Therefore formula (4) is fulfilled.

The proof is completed.

3. PERIODIC SOLUTIONS OF PERTURBED SYSTEM

Let $f \in \mathbf{C}^\infty(\mathbf{R}^n, \mathbf{R}^n)$. We consider the system

$$\dot{x} = f(x), \qquad x \in \mathbf{R}^n \tag{13}$$

and its perturbed analog

$$\dot{x} = f(x) + g(x, \epsilon), \qquad \epsilon \in \mathbf{R}^m \tag{14}$$

where $g \in \mathbf{C}^\infty(\mathbf{R}^n \times \mathbf{R}^m, \mathbf{R}^n)$.

First we introduce the general definition of structurally stable periodic solutions of the system (13).

Definition 1. Let $\mathcal{M}$ be a subset in $\mathbf{C}^\infty(\mathbf{R}^n \times \mathbf{R}^m, \mathbf{R}^n)$ and let the zero function be a limit point of $\mathcal{M}$. We shall say that the periodic solution $x = \bar{x}(t)$ of system (13) is *$\mathcal{M}$-structurally stable* if there exists a neighborhood $\mathcal{U}$ of zero in $\mathbf{C}^\infty(\mathbf{R}^n \times \mathbf{R}^m, \mathbf{R}^n)$ such that the system

$$\dot{x} = f(x) + g(x, \epsilon) \tag{15}$$

has periodic solution $x = \bar{x}_\epsilon(t)$ for all $g \in \mathcal{U} \cap \mathcal{M}$ and

$$\lim_{\|g\| \to 0, g \in \mathcal{U} \cap \mathcal{M}} \|\bar{x}(t) - \bar{x}_\epsilon(t)\| = 0 \tag{16}$$

Clearly, if $\mathcal{M} = \mathbf{C}^\infty(\mathbf{R}^n \times \mathbf{R}^m, \mathbf{R}^n)$ then we obtain the classical statement of the problem for the existence of periodic solution of the perturbed system (14). If

$$\mathcal{M} = \{g = (g_1, \ldots, g_n) \in C^\infty(R^n \times R^m, R^n):$$
$$g_k(x, \epsilon) = \cdots = g_n(x, \epsilon) = 0 \quad \text{for some } k \in \{1, \ldots, n\}\}$$

then we obtain a statement for the existence of periodic solution under "small" deviation of some equations of system (14).

In view of Definition 1, the basic question is: what is the "maximal" set $\mathcal{M}$ such that the system (13) has $\mathcal{M}$-structurally stable periodic solution? Here the word "maximal" means that if $\mathcal{M}_1$ is a subset in $C^\infty(R^n \times R^m, R^n)$ such that the system (13) has $\mathcal{M}_1$-structurally stable periodic solution, then $\mathcal{M}_1 \subseteq \mathcal{M}$.

Unfortunately, the complicated structure of the "maximal" set $\mathcal{M}$ under consideration in Definition 1 impedes our intention to establish some results about the perturbed term g in system (14) (see "center-focus" problem). That is why we shall consider the case when the set $\mathcal{M}$ has "linear-like" structure.

We introduce the following hypotheses:

(H1.1) All solutions of systems (13) and (14) are C^∞-smooth and their maximal interval of existence and uniqueness is R.
(H1.2) $g(x, 0) = 0$, $x \in R^n$.
(H1.3) The system (13) has an ω-periodic solution $x = \bar{x}(t)$, $\omega > 0$.

Definition 2. Let hypotheses (H1) hold. We shall say that the periodic solution $x = \bar{x}(t)$ of system (13) is *(g, d)-structurally stable* if there exist a neighborhood U of zero in R^d, $d \leq m$, and a C^∞-smooth map $\epsilon: U \to R^m$ such that $\epsilon(0) = 0$, the system

$$\dot{x} = f(x) + g(x, \epsilon(\nu)) \tag{17}$$

has periodic solution $x = \bar{x}_\nu(t)$ for all $\nu \in U$, and

$$\lim_{\|\nu\| \to 0} \|\bar{x}(t) - \bar{x}_\nu(t)\| = 0 \tag{18}$$

Remark 1. It is not difficult to see that for every system (13) there exists a function $g = g(x, \epsilon)$ such that the ω-periodic solution $x = \bar{x}(t)$ of (13) is (g, n)-structurally stable. Indeed, if we set $g(x, \epsilon) = f(x - \epsilon) - f(x)$, $\epsilon \in R^n$, then the system (17) has ω-periodic solution $\bar{x}_\epsilon(t) = \bar{x}(t) + \epsilon$.

Obviously, there exist functions $f = f(x)$ and $g = g(x, \epsilon)$ such that the system (13) has periodic solution and the system (14) does not have any periodic solution.

Remark 2. If for each $g \in C^\infty(R^n \times R, R^n)$ the periodic solution $x = \bar{x}(t)$ of system (13) is $(g, 1)$-structurally stable, we obtain the classical statement of the problem for the existence of periodic solution of the perturbed system (14).

Let $A(t) = \mathrm{D}_x f(x)|_{x=\bar{x}(t)}$ and $A^*(t)$ be the transposition of $A(t)$. We consider the system

$$\dot{x} = A(t)x, \qquad x \in \mathrm{R}^n \tag{19}$$

and its conjugate system

$$\dot{\psi} = -A^*(t)\psi, \qquad \psi \in \mathrm{R}^n \tag{20}$$

Let $\{\tilde{\psi}_1(t), \ldots, \tilde{\psi}_d(t)\}$, $d \leq n$, be a basis for the space of all periodic solutions of (20). We set

$$b_{ij} = \int_0^\omega \left\langle \tilde{\psi}_i(t), \frac{\partial g}{\partial \epsilon_j}(\bar{x}(t), 0) \right\rangle dt \tag{21}$$

where $i \in \{1, \ldots, d\}, j \in \{1, \ldots, m\}$; $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in R^n; $B = (b_{ij})$ denotes the $d \times m$ matrix with elements (21). Let B_k be the matrix with elements b_{ij}, $i \in \{1, \ldots, k\}, j \in \{1, \ldots, k\}$, $k \leq d$.

Theorem 1. Let hypotheses (H1) be valid and let rank $B = d$. Then the periodic solution $x = \bar{x}(t)$ of system (13) is (g, d)-structurally stable. Moreover, there exists a number $r_0 > 0$ such that if $x = \bar{x}_\nu(t)$ is a periodic solution of the system (17) and

$$\{\bar{x}_\nu(t): t \in \mathrm{R}\} \subset \bigcup_{t \in [0,\omega)} \{\mathrm{B}_{R^n}(\bar{x}(t), r_0)\}$$

then the period of $x = \bar{x}_\nu(t)$ is ω.

Proof. Let $x = x(t)$ be a solution of (13). We set

$$y(t) = x(t) - \bar{x}(t)$$

$$\mathcal{X} = \{y \in \mathrm{C}^\infty(\mathrm{R}, \mathrm{R}^n): y(t + \omega) = y(t), t \in \mathrm{R}\}, \qquad \mathcal{Y} = \mathrm{R}^m$$

$$F: \mathcal{X} \times \mathrm{R}^m \to \mathcal{X}, \qquad F(y, \epsilon) = \dot{y} - f(y + \bar{x}(t)) + f(\bar{x}(t)) - g(y + \bar{x}(t), \epsilon)$$

The set $\mathcal{X}$ with scalar product $(y_1, y_2) = \int_0^\omega \langle y_1(t), y_2(t) \rangle\, dt$, $y_1, y_2 \in \mathcal{X}$, and corresponding norm is a Hilbert space.

We shall consider the equation

$$F(y, \epsilon) = 0 \tag{22}$$

First, we shall verify conditions 1 and 2 of Lemma 1.

1. Obviously, F is a C^∞-smooth map; if $y(t) = 0$, $t \in \mathrm{R}$, and $\epsilon = 0$, then $F(0, 0) = 0$.

2. From the definition of the map F, it follows that

$$\mathbf{D}_y F(0, 0)y = \dot{y} - \mathbf{D}_x f(\bar{x}(t))y - \mathbf{D}_x g(\bar{x}(t), 0)y = \dot{y} - A(t)y \tag{23}$$
$$\mathbf{D}_y F(0, 0)\epsilon = -\mathbf{D}_x g(\bar{x}(t), 0)\epsilon$$

Without loss of generality, we suppose that $\det B_d \neq 0$.

Let $\epsilon \in \mathbf{R}^m$, $\epsilon = (\epsilon^1, \epsilon^2)$, $\epsilon^1 = (\epsilon_1, \ldots, \epsilon_d) \in \mathbf{R}^d$, $\epsilon^2 = (\epsilon_{d+1}, \ldots, \epsilon_m) \in \mathbf{R}^{m-d}$, $\mathcal{W} = \{(\epsilon^1, 0) \in \mathbf{R}^m \colon \epsilon^1 \in \mathbf{R}^d\}$.

First, we shall prove that $\operatorname{im} \mathbf{D}_y F(0, 0) \cap \operatorname{im} \mathbf{D}_\epsilon F(0, 0)|_{\mathcal{W}} = \{0\}$.

Indeed, let us suppose that there exists $\epsilon \in \mathcal{W}\setminus\{0\}$ such that $\mathbf{D}_\epsilon F(0, 0)\epsilon \in \operatorname{im} \mathbf{D}_y F(0, 0)$. Then there exists $\eta \in \mathcal{X}\setminus\{0\}$ such that $\mathbf{D}_\epsilon F(0, 0)\epsilon = \mathbf{D}_y F(0, 0)\eta$, i.e., $\eta = \eta(t)$ is a solution of the system

$$\dot{\eta} - A(t)\eta = -\mathbf{D}_\epsilon g(\bar{x}(t), 0)\epsilon \tag{24}$$

From Hartman (1964, Chapter XII, Theorem 1.2), it follows that the system (24) has a solution if and only if

$$\int_0^\omega \langle \tilde{\psi}_i(t), \mathbf{D}_\epsilon g(\bar{x}(t), 0)\epsilon \rangle \, dt = 0 \tag{25}$$

for all $i \in \{1, \ldots, d\}$.

Therefore

$$\begin{aligned} 0 &= \int_0^\omega \langle \tilde{\psi}_i(t), \mathbf{D}_\epsilon g(\bar{x}(t), 0)\epsilon \rangle \, dt \\ &= \sum_{k=1}^n \int_0^\omega \tilde{\psi}_{ik}(t) \sum_{j=1}^m \frac{\partial g_k(\bar{x}(t), 0)}{\partial \epsilon_j} \epsilon_j \, dt \\ &= \sum_{j=1}^m \epsilon_j \int_0^\omega \left\langle \tilde{\psi}_i(t), \frac{\partial g(\bar{x}(t), 0)}{\partial \epsilon_j} \right\rangle dt \\ &= \sum_{j=1}^d b_{ij}\epsilon_j + \sum_{j=d+1}^m b_{ij}\epsilon_j \end{aligned}$$

where $\tilde{\psi}_i(t) = (\tilde{\psi}_{i1}(t), \ldots, \tilde{\psi}_{in}(t))$ and $g(x, \epsilon) = (g_1(x, \epsilon), \ldots, g_n(x, \epsilon))$. From the definition of the space $\mathcal{W}$ it follows that $\epsilon_j = 0$, $j \in \{d + 1, \ldots, m\}$. Hence $\sum_{j=1}^d b_{ij}\epsilon_j = 0$. The obtained equality contradicts $\det B_d \neq 0$. Therefore, if $\epsilon \in \mathcal{W}\setminus\{0\}$, then $\mathbf{D}_\epsilon F(0, 0)\epsilon \notin \operatorname{im} \mathbf{D}_y F(0, 0)$.

From (23) it follows that $\ker \mathbf{D}_y F(0, 0)$ consists of all ω-periodic solutions of the system $\dot{y} = A(t)y$, i.e., $\dim \ker \mathbf{D}_y F(0, 0) = d = \dim \mathcal{W}$. Hence formula (3) is true.

We shall prove the Fredholm condition (12). From Hartman (1964, Chapter XII, Theorem 1.1) it follows that the system $\dot{y} - A(t)y = 0$, $y \in \mathcal{X}$, has a solution if and only if the system $\dot{y} - A(t)y = h(t)$, $h \in \mathcal{X}\setminus\operatorname{im} \mathbf{D}_y F(0,$

0), has no solution $y \in \mathcal{X}$. But the system $\dot{y} - A(t)y = 0$ has exactly d linearly independence solutions in space $\mathcal{X}$. Therefore, dim coker $\mathbf{D}_y F(0, 0) = d$, i.e., the formula (12) is valid.

Hence, condition 2 of Lemma 1 follows from Lemma 2.

Therefore, there exist a number $r_1 > 0$ and unique smooth maps y: $\mathrm{B}_{\mathrm{R}^{m-d}}(0, r_1) \to \mathcal{X}$, $\nu \to y_\nu$, and ϵ: $\mathrm{B}_{\mathrm{R}^{m-d}}(0, r_1) \to \mathrm{R}^m$, $\nu \to \epsilon_\nu$, such that $y_0 = 0$, $\epsilon_0 = 0$, and

$$F(y_\nu, \epsilon_\nu) = 0 \qquad \text{for any} \quad \nu \in \mathrm{B}_{\mathrm{R}^{m-d}}(0, r_1)$$

Thus

$$\dot{y}_\nu(t) - f(y_\nu(t) + \bar{x}(t)) + f(\bar{x}(t)) - g(y_\nu(t) + \bar{x}(t), \epsilon_\nu) = 0$$

Writing $x_\nu(t) = y_\nu(t) + \bar{x}(t)$, we obtain that

$$\dot{x}_\nu(t) = f(x_\nu(t)) + g(x_\nu(t), \epsilon_\nu)$$

i.e., the function $x_\nu \in \mathcal{X}$ is a solution of the perturbed system (17) for any $\nu \in \mathrm{B}_{\mathrm{R}^{m-d}}(0, r_1)$.

Theorem 1 is proved.

Let

$$b_{im+1} = \int_0^\omega \langle \tilde{\psi}_i(t), \dot{\bar{x}}(t) \rangle \, dt \tag{26}$$

where $i \in \{1, \ldots, d\}$, $B_0 = (b_{ij})$ denotes the $d \times (m + 1)$ matrix with elements (21) and (26).

Theorem 2. Let hypotheses (H1) hold and let rank $B_0 = d$. Then the periodic solution $x = \bar{x}(t)$ of the system (13) is (g, d)-structurally stable.

Proof. The proof of Theorem 2 is similar to the proof of Theorem 1, except for the choice of the function F. Thus we sketch only some steps of the proof.

We set

$$y(t, \alpha) = x(t + \alpha t) - \bar{x}(t), \qquad \alpha \in \mathrm{R}$$

$$\mathcal{X} = \{y \in \mathrm{C}^\infty(\mathrm{R}^2, \mathrm{R}^n): y(t + \omega, \alpha) = y(t, \alpha), t \in \mathrm{R}\}, \qquad \mathcal{Y} = \mathrm{R}^m$$

$$F: \quad \mathcal{X} \times \mathrm{R}^{1+m} \to \mathrm{R}^n$$

$$F(y, (\alpha, \epsilon)) = \dot{y}(t, \alpha) - f(y(t, \alpha) + \bar{x}(t)) + f(\bar{x}(t)) - g(y(t, \alpha) + \bar{x}(t), \epsilon)$$

where $x = x(t)$ is a solution of (13) and the set $\mathcal{X}$ with scalar product $(y_1, y_2) = \int_0^\omega \langle y_1(t), y_2(t) \rangle \, dt$, $y_1, y_2 \in \mathcal{X}$, and the corresponding norm is Hilbert space.

Then:

1. If $\alpha = 0$, $y(t) = 0$, $t \in \mathbf{R}$, and $\epsilon = 0$, then $F(0, (0, 0)) = 0$.
2. From the definition of function F it follows that

$$\mathbf{D}_y F(0, (0, 0)) = \dot{y} - \mathbf{D}_x f(\bar{x}(t))y - \mathbf{D}_x g(\bar{x}(t), 0)y = \dot{y} - A(t)y$$

Obviously

$$\mathbf{D}_\alpha y(t, \alpha)_{|y(t,0)=0} = t\dot{\bar{x}}(t)$$

because if $y(t, 0) = 0$, then $x(t) = \bar{x}(t)$. Analogously, we have

$$\mathbf{D}_\alpha \dot{y}(t, \alpha)_{|y(t,0)=0} = \dot{\bar{x}}(t) + t\ddot{\bar{x}}(t)$$

Therefore

$$\begin{aligned}
&\mathbf{D}_{(\alpha,\epsilon)} F(0, (0, 0))(\alpha, \epsilon) \\
&\quad = \mathbf{D}_\alpha \dot{y}(t, \alpha)_{|y(t,0)=0} - \mathbf{D}_x f(\bar{x}(t))\mathbf{D}_\alpha y(t, \alpha)_{|y(t,0)=0} - \mathbf{D}_\epsilon g(\bar{x}(t), 0)\epsilon \\
&\quad = \dot{\bar{x}}(t) + t\ddot{\bar{x}}(t) - \mathbf{D}_x f(\bar{x}(t))t\dot{\bar{x}}(t) - \mathbf{D}_\epsilon g(\bar{x}(t), 0)\epsilon \\
&\quad = \dot{\bar{x}}(t) - \mathbf{D}_\epsilon g(\bar{x}(t), 0)\epsilon
\end{aligned}$$

because $x = \bar{x}(t)$ is a solution of (13), i.e., $\ddot{\bar{x}}(t) - \mathbf{D}_x f(\bar{x}(t))\dot{\bar{x}}(t)$.

The assumption $\mathbf{D}_{(\alpha,\epsilon)} F(0, (0, 0))(\alpha, \epsilon) \in \operatorname{im} \mathbf{D}_y F(0, 0)$ for some $(\alpha, \epsilon) \in \mathbf{R}^{1+m}$ is equivalent to the existence of $\eta \in \mathcal{X}$ such that

$$\dot{\eta} - A(t)\eta = \dot{\bar{x}}(t) - \mathbf{D}_\epsilon g(\bar{x}(t), 0)\epsilon \tag{27}$$

The system (27) has a solution $\eta = \eta_0(t) \in \mathcal{X}$ if and only if

$$\int_0^\omega \langle \tilde{\psi}_i(t), \dot{\bar{x}}(t) - \mathbf{D}_\epsilon g(\bar{x}(t), 0)\epsilon \rangle \, dt = 0, \qquad i \in \{1, \ldots, d\} \tag{28}$$

From the formula (28) and condition rank $B_0 = d$ it follows that there exists a d-dimensional space $\mathcal{W} \subset \mathbf{R}^{1+m}$ such that $\operatorname{im} \mathbf{D}_y F(0, (0, 0)) \cap \operatorname{im} \mathbf{D}_{(\alpha,\epsilon)} F(0, (0, 0))|_{\mathcal{W}} = \{0\}$.

From $\mathbf{D}_y F(0, (0, 0)) = \dot{y} - A(t)y$, it follows $\dim \ker \mathbf{D}_y F(0, 0) = d = \dim \mathcal{W}$. Hence formula (3) is true.

The proof of the Fredholm condition (12) is analogous to the corresponding part of the proof of Theorem 1.

Therefore, there exist a number $r_1 > 0$ and unique smooth maps y: $\mathbf{B}_{\mathbf{R}^{m-d}}(0, r_1) \to \mathcal{X}$, $v \to y_v$, ϵ: $\mathbf{B}_{\mathbf{R}^{m-d}}(0, r_1) \to \mathbf{R}^m$, $v \to \epsilon_v$, and α: $\mathbf{B}_{\mathbf{R}^{m-d}}(0, r_1) \to \mathbf{R}$, $v \to \alpha_v$, such that $y_0 = 0$, $\epsilon_0 = 0$, $\alpha_0 = 0$, and

$$F(y_v, (\epsilon_v, \alpha_v)) = 0 \qquad \text{for any} \quad v \in \mathbf{B}_{\mathbf{R}^{m-d}}(0, r_1)$$

Thus

$$\dot{y}_v(t, \alpha_v) - f(y_v(t, \alpha_v) + \bar{x}(t)) + f(\bar{x}(t)) - g(y_v(t, \alpha_v) + \bar{x}(t), \epsilon_v) = 0$$

Writing $x_\nu(t) = y_\nu(t, \alpha_\nu) - \bar{x}(t)$, we obtain that

$$\dot{x}_\nu(t) = f(x_\nu(t)) + g(x_\nu(t), \epsilon_\nu)$$

i.e., the function $x_\nu \in \mathscr{X}$ is a solution of the perturbed system (17) for any $\nu \in B_{R^{m-d}}(0, r_1)$.

This completes the proof.

Example 1. Consider the following system (Nemitsky and Stepanov, 1949, Chapter II, §5):

$$\begin{aligned} \dot{x}_1 &= P(x_1, x_2) = -x_2 + x_1(x_1^2 + x_2^2 - 1)^{2p+1} \\ \dot{x}_2 &= Q(x_1, x_2) = x_1 + x_2(x_1^2 + x_2^2 - 1)^{2p+1} \end{aligned} \tag{29}$$

and its perturbed analog

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1(x_1^2 + x_2^2 - 1)^{2p+1} + g_1(x_1, x_2, \epsilon) \\ \dot{x}_2 &= x_1 + x_2(x_1^2 + x_2^2 - 1)^{2p+1} + g_2(x_1, x_2, \epsilon) \end{aligned} \tag{30}$$

where $x = (x_1, x_2)$, $p \geq 1$, the function $g = (g_1, g_2)$ satisfies hypotheses (H1), and $\epsilon \in R$.

Obviously, the system (29) has a unique periodic solution $\{(x_1, x_2)\colon x_1 = \cos t, x_2 = \sin t, t \in R\}$.

Let

$$\begin{aligned} A(t) &= \frac{\partial(P(x_1, x_2), Q(x_1, x_2))}{\partial(x_1, x_2)}\bigg|_{\substack{x_1=\cos t\\ x_2=\sin t}} \\ &= \begin{bmatrix} (x_1^2 + x_2^2 - 1)^{2p}[(4p+3)x_1^2 + x_2^2 - 1] & -1 + 2(2p+1)x_1x_2(x_1^2 + x_2^2 - 1)^{2p} \\ 1 + 2(2p+1)x_1x_2(x_1^2 + x_2^2 - 1)^{2p} & (x_1^2 + x_2^2 - 1)^{2p}[x_1^2 + (4p+3)x_2^2 - 1] \end{bmatrix}_{\substack{x_1=\cos t\\ x_2=\sin t}} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Therefore, $\{\tilde{\psi}_1, \tilde{\psi}_2\}$, where $\tilde{\psi}_1 = (\cos t, \sin t)$ and $\tilde{\psi}_2 = (-\sin t, \cos t)$, is a basis for the system $\dot{\psi} = A(t)\psi$, $\psi \in R^2$.

Thus if

$$b_{i1} = \int_0^{2\pi} \left\langle \tilde{\psi}_i, \left(\frac{\partial g_1}{\partial \epsilon}(\cos t, \sin t, 0), \frac{\partial g_2}{\partial \epsilon}(\cos t, \sin t, 0)\right)\right\rangle dt \neq 0,$$

$i = 1$ or 2

then the system (30) has a 2π-periodic solution (see Theorem 1).

For example, if $g_1(x_1, x_2, \epsilon) = \epsilon x_1$, $g_2(x_1, x_2, \epsilon) = \epsilon x_2$, then the system (30) has a 2π-periodic solution.

Example 2. Consider the two-dimensional system

$$\begin{aligned} \dot{x}_1 &= ax_2 + \epsilon g_1(x_1, x_2) \\ \dot{x}_2 &= -ax_1 + \epsilon g_2(x_1, x_2) \end{aligned} \tag{31}$$

where $a, \epsilon \in \mathrm{R}$, $a \neq 0$, $g_1, g_2 \in \mathrm{C}(\mathrm{R}^2, \mathrm{R})$.

A classical result due to Poincaré states that if the functions g_1 and g_2 satisfy the symmetry condition

$$g_1(-x_1, x_2) = -g_1(x_2, x_2), \qquad g_2(-x_1, x_2) = g_2(x_1, x_2) \tag{32}$$

then (31) has periodic solution with period near $2\pi/|a|$ for sufficiently small ϵ. In the present example, we shall apply Theorem 1 to prove that if equations (32) are valid and if

$$\int_0^{2\pi/|a|} g_1(\sin at, \cos at)\sin at + g_2(\sin at, \cos at)\cos at\, dt \neq 0$$

then (31) has $2\pi/|a|$-periodic solution for sufficiently small ϵ.

The system $\dot{x}_1 = ax_2$, $\dot{x}_2 = -ax_1$ has $2\pi/|a|$-periodic solution $\bar{x}_1(t) = \sin at$, $\bar{x}_2(t) = \cos at$. Moreover, $\{\tilde{\psi}_1, \tilde{\psi}_2\}$, where $\tilde{\psi}_1 = (\sin at, \cos at)$, $\tilde{\psi}_2 = (-\cos at, \sin at)$ is a basis for the space of all solutions of the system $\dot{\psi} = A(t)\psi$, $\psi \in \mathrm{R}^2$, $A(t) = \left[\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right]$. Let us calculate

$$\begin{aligned} b_{11} &= \int_0^{2\pi/|a|} \langle \tilde{\psi}_1(t), (g_1(\bar{x}_1(t), \bar{x}_2(t)), g_2(\bar{x}_1(t), \bar{x}_2(t)))\rangle\, dt \\ &= \int_{-\pi/|a|}^{\pi/|a|} g_1(\sin at, \cos at)\sin at + g_2(\sin at, \cos at)\cos at\, dt \\ &= 2\int_0^{\pi/|a|} g_1(\sin at, \cos at)\sin at + g_2(\sin at, \cos at)\cos at\, dt \neq 0 \end{aligned}$$

Therefore (see Theorem 1), there exists $r_1 > 0$ such that if $|\epsilon| < r_1$, then the system (31) has $2\pi/|a|$-periodic solution.

Let us remark that Theorem 2 cannot apply to the system (31), because rank $B_0 = 1$, $\min\{d, m + 1\} = 2$.

Before giving some particular cases of Theorems 1 and 2 we shall prove the following lemma.

Lemma 3. Let hypotheses (H1.1) and (H1.2) be valid. Then the following two statements are equivalent:

1. $\int_0^\omega \langle \tilde{\psi}_i(t), \dot{\bar{x}}(t) \rangle \, dt \neq 0$ for some $i \in \{1, \ldots, d\}$.
2. 1 is a simple characteristic multiplier for (19).

Proof. In Hartman (1964, Chapter XII, Lemma 1.1) it is proved that the system (13) has nontrivial periodic solution $x = \bar{x}(t)$ if and only if 1 is a characteristic multiplier for (19).

Let us suppose that 1 is a nonsimple characteristic multiplier for (19). Then there exist matrices $A_1(t)$, $A_2(t)$, and $A_3(t)$ such that:

1. $A_1(t)$ is a nonsingular periodic matrix. The last column of $A_1(t)$ is $\dot{\bar{x}}(t)$.
2. $A_2(t)$ is a nonsingular matrix.
3. If $X(t)$ is a fundamental matrix of (19), then

$$X(t) = A_1(t) \begin{bmatrix} A_2(t) & \mathbf{0} \\ A_3(t) & \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \end{bmatrix}$$

where $\mathbf{0}$ is a zero matrix with proper dimension.

Then

$$\dot{X}(t) = \dot{A}_1(t) \begin{bmatrix} A_2(t) & \mathbf{0} \\ A_3(t) & \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \end{bmatrix} + A_1(t) \frac{d}{dt} \begin{bmatrix} A_2(t) & \mathbf{0} \\ A_3(t) & \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \end{bmatrix}$$

or

$$\dot{A}_1(t) = \left[\dot{X}(t) - A_1(t) \begin{bmatrix} \dot{A}_2(t) & \mathbf{0} \\ \dot{A}_3(t) & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \right] \begin{bmatrix} A_2(t) & \mathbf{0} \\ A_3(t) & \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \end{bmatrix}^{-1}$$

But $\dot{X}(t) = A(t)X(t)$ and

$$\begin{bmatrix} A_2(t) & \mathbf{0} \\ A_3(t) & \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \end{bmatrix}^{-1} = \begin{bmatrix} A_2^{-1}(t) & \mathbf{0} \\ A_4(t) & \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \end{bmatrix}$$

where

$$A_4(t) = \begin{bmatrix} -1 & 0 \\ t & -1 \end{bmatrix} A_3(t) A_2^{-1}(t)$$

(Horn and Johnson, 1986, §0.7.3).

Therefore

$$\dot{A}_1(t) = A(t)A_1(t) - A_1(t)\begin{bmatrix} \dot{A}_2(t) & \mathbf{0} \\ \dot{A}_3(t) & \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \end{bmatrix}\begin{bmatrix} A_2^{-1}(t) & \mathbf{0} \\ A_4(t) & \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \end{bmatrix}$$

Thus the $(n - 1)$th column $a_{n-1}(t)$ of $A_1(t)$ satisfies

$$\dot{a}_{n-1}(t) = A(t)a_{n-1}(t) - \dot{\bar{x}}(t)$$

Consequently (Hartman, 1964, Chapter XII, Theorem 1.2)

$$\int_0^\omega \langle \tilde{\psi}_i(t), \dot{\bar{x}}(t) \rangle \, dt = 0, \qquad i \in \{1, \ldots, d\} \tag{33}$$

Equation (33) contradicts assumption 1 of the lemma.
Lemma 3 is proved.

Corollary 1. Let the hypotheses (H1) be valid and let

$$\int_0^\omega \langle \tilde{\psi}_i(t), \dot{\bar{x}}(t) \rangle \, dt \neq 0 \qquad \text{for some} \quad i \in \{1, \ldots, d\}$$

Then the periodic solution $x = \bar{x}(t)$ of system (13) is structurally stable, i.e., for any $g \in \mathbf{C}^\infty(\mathrm{R}^n, \mathrm{R}^n)$, $g(x, 0) = 0$, $(\partial g/\partial x)(x, 0) = 0$ there exists $\delta > 0$ such that the system

$$\dot{x} = f(x) + g(x, \nu), \qquad \nu \in (-\delta, \delta)$$

has a periodic solution $x = \bar{x}_\nu(t)$ and

$$\lim_{\|\nu\| \to 0} \|\bar{x}(t) - \bar{x}_\nu(t)\| = 0$$

The proof of Corollary 1 follows from Theorem 2.

Corollary 2. Let the hypotheses (H1) be satisfied and let 1 be a simple characteristic multiplier for (19). Then the periodic solution $x = \bar{x}(t)$ of the system (13) is structurally stable.

The proof of Corollary 2 follows from Theorem 2 and Lemma 3.

Remark 3. One may find theorems analogous to Corollary 2 in Hartman (1964, Chapter XII, Theorem 2.2) or Rouche *et al.* (1977), where the proof is based on the classical Implicit Function Theorem. Thus Theorems 1 and 2 generalized cited results.

Remark 4. The following well-known result immediately follows from Corollary 2. If the periodic solution $x = \bar{x}(t)$ of the system (13) is hyperbolic, then it is structurally stable [for the definition of a hyperbolic periodic orbit of an autonomous system see Palis and De Melo (1982, Chapter 3, §§1.2–1.3)].

ACKNOWLEDGMENTS

I owe special gratitude to Prof. A. Dishliev for his careful reading of the manuscript. This research was partially supported by the Ministry of Education, Science and Technologies under grant MM511.

REFERENCES

Coddington, E. A., and Levinson, N. (1955). *Theory of Ordinary Differential Equations,* McGraw-Hill, New York.

Hartman, P. (1964). *Ordinary Differential Equations,* Wiley, New York.

Horn, R. A., and Johnson, C. R. (1986). *Matrix Analysis,* Cambridge University Press, Cambridge.

Hutson, V. C. L., and Pym, J. S. (1980). *Applications of Functional Analysis and Operator Theory,* Academic Press, New York.

Kirilov, A. A., and Gvishiany, A. D. (1979). *Theorems and Examples in Functional Analysis,* Nauka, Moscow.

Krein, S. G. (1967). *Linear Differential Equations in Banach Spaces,* Nauka, Moscow.

Li Bingxi (1981). Periodic orbits of autonomous ordinary differential equations: Theory and applications, *Nonlinear Analysis, Theory, Methods and Applications,* **5**(9), 931–958.

Massera, J. L. (1950). The existence of periodic solutions of systems of differential equations, *Duke Mathematical Journal,* **17**, 457–475.

Nemitsky, V. V., and Stepanov, V. V. (1949). *Qualitative Theory of Ordinary Differential Equations,* 2nd ed., Moscow.

Nitecki, Z. (1971). *Differentiable Dynamics. An Introduction to the Orbit Structure of Diffeomorphisms,* MIT Press, Cambridge, Massachusetts.

Palis, J., and De Melo, W. (1982). *Geometric Theory of Dynamical Systems. An Introduction,* Springer-Verlag, Berlin.

Rouche, N., Habets, P., and Laloy, M. (1977). *Stability Theory by Liapunov's Direct Method,* Springer-Verlag, Berlin.

Yoshizawa, T. (1966). *Stability Theory by Liapunov's Second Method,* Mathematical Society of Japan.